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PRELIMINARY NOTE CONCERNING A NEW THEORY OF THE MOTIONS OF THE STARS

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Since Kapteyn's discovery of the star streams a great many investigators have taken up the study of the systematic motions of the stars. The original line of attack, originated by Charlier, was to compute from given rectangular velocity components $\dot{X}$, $\dot{Y}$, $\dot{Z}$, the Jacobean ellipsoid, the directions of the three axes of this ellipsoid and the velocity dispersions along the axes. The major axis of the velocity ellipsoid gives the direction of Kapteyn's star streams. The question whether physically Kapteyn's conception of distinct streams or Schwarzschild's interpretation of an ellipsoidal distribution is correct has never been settled satisfactorily. For the applicability of Charlier's method this is of no consequence. It is, however, not *a priori* evident that the direction of preferential motion derived by this method is physically the most significant direction for an understanding of the stellar velocities. This may be exemplified by the fictitious case where we have a distribution of points that follow a sine-curve; the preferential direction as computed from such a scatter diagram will in general not be the same as the direction of propagation of the sine-curve. In the case of the distribution of the stellar velocities it is easy to point out marked regular features which cannot be accounted for by an ellipsoidal distribution, i.e., a Maxwellian distribution whereby the $\dot{X}$, $\dot{Y}$ and Z have different moduli of precision. This can be shown in spite of the fact that the velocities have considerable errors due largely to the errors in the parallaxes of the stars. In fact it is one of the interesting results of the present investigation that the stellar parallaxes appear to have greater accuracy than has been thought by several statistical workers if not by the parallax observers themselves.

The stars most suitable for a detailed study of their velocities are doubtless the ones of spectral type A. It is well known that the velocity dispersion of the early type stars is less than for the late type stars, and also that the B stars have parallaxes that are mostly too small to expect accu-

rate velocities. It is therefore logical to begin with the A stars and to investigate later on to what extent the other stars follow the laws of motion of the A stars and in which respect they deviate.

I have used the rectangular equatorial velocity components for the A stars using the parallaxes as given in the Yale Catalogue of Bright Stars.

If we consider the velocity figure, where each star is represented by a point, $\dot{X}\ \dot{Y}\ \dot{Z}$, we want to find a plane in this figure such that the gradient of the density of the points $\dot{X}\ \dot{Y}\ \dot{Z}$ in the direction at right angles to this plane is a maximum. This has been found in the following way. Take a preliminary direction $\dot{Z}_1$ and compute the components $\dot{Z}_1$ and also the components in two rectangular directions $\dot{X}_1$ and $\dot{Y}_1$. Plotting $\dot{Z}_1$ against $\dot{X}_1$ and $\dot{Y}_1$, respectively, we find the angles through which our coördinate system has to be rotated in order to improve the $\dot{Z}_1$-direction. By successive approximation the direction $\dot{Z}$ has been found to be in galactic longitude 50°.

The frequency distribution of $\dot{Z}$ now has two maxima, at -22 and at 0 km./sec. Considering the inevitable errors of observations the true velocities $\dot{Z}$ can be assumed to be either -22 or 0. The first group, T, includes the Hyades; the second group, S, includes the Ursa Major stars and the sun. This distribution shows as far as I can see either that the stars belong to one of two discrete groups or that the distribution of the $\dot{Z}$ components follows the law of the frequency of the sine. In either case we get a frequency curve with two maxima.

The $\dot{Z}$ components furthermore depend on the positions of the stars in space. This follows from a diagram where $\dot{Z}$ is plotted as ordinates against Z as abscissae. The exact form of $\dot{Z}$ as function of Z requires further study. The present material of A stars indicates a linear relation, the rate of increase in $\dot{Z}$ being about 5 km./sec. per 100 parsecs in the direction of the positive Z axis (toward galactic longitude 50°). The A stars of parallaxes larger than $0''.020$ show an even greater rate of expansion in the Z direction, and originally I have adopted a statistical value of 10 km./sec. per 100 parsecs. Putting the above results in a formula we have

$$V \cos \lambda + \frac{\mu_\lambda \sin \lambda}{\pi} 4.74 = a + \frac{0.1 \cos \lambda}{\pi}$$

where V is the radial velocity, λ the angle between the direction of the star and the positive Z axis, μ_λ the component of the proper motion in the direction toward the apex of the positive Z axis and a is either 0 or -22. We then have

$$\pi = \frac{4.74\mu_\lambda \sin \lambda - 0.1 \cos \lambda}{a - V \cos \lambda}.$$

By means of this formula I have computed parallaxes for a great many

stars. Wholesale statistical parallaxes that are not based on the magnitudes would be of some importance even if the individual accuracy were small. It appears, however, that the values derived by the present formula agree exceedingly well with the trigonometric parallaxes.

The components $\dot{Z}$ have been computed for all stars whose parallaxes are not less than $0''.050$ in the Catalogue of Bright Stars. Whereas $\dot{Z}$ is restricted to the values 0 and -22 for stars of spectra earlier than *F*8 it appears that although the two maxima in the frequency distribution of $\dot{Z}$ are still present for the later type stars, a large part belong to other groups, extending beyond the interval of 0 to -22 km./sec. The fact that the early type stars have no representatives among these latter groups is presumably local, because, if we compute $\dot{Z}$ for stars of smaller parallaxes, there appear several for which $\dot{Z}$ is outside the range of my groups *S* and *T*. In using as I did stars of large trigonometric parallaxes, more than a due share of such stars are found among the late type stars.

Parallaxes have been computed by means of the above formula for different samples of stars. A few difficulties have to be considered. The dual value of the constant a in the formula would in general give two values for each parallax. If both are negative, my formula fails: the velocity of the star is either erroneous or "stray." If one value is positive and the other negative, I take the positive value and say that the parallax is non-ambiguous. If both are positive we would have to use some judgment based on *a priori* knowledge, such as spectral type and magnitude. For a first test, however, we leave these stars out and confine ourselves to the non-ambiguous cases. If the denominator is less than 5 I have rejected the determination for the reason of low weight. Finally I have in the first sample omitted all stars for which the formula gave less than 0.100 in the numerator. This last restriction which excludes the stars of small proper motions has already been found to be unnecessarily severe.

As a first sample I have computed the parallaxes for the stars in Schlesinger's first Catalogue of Parallaxes. The computed values represent the observed parallaxes very satisfactorily for the stars of spectral types not later than *F*2. The later type stars have large discrepancies in many cases. This was to be expected from the fact that among the later type stars several have $\dot{Z}$ velocities outside the range of my *T* and *S* streams, especially since most of the faint stars have been observed on account of large proper motions. If we confine ourselves to stars not fainter than 5.5 the influence of this selection is largely eliminated and we then find that 50% of the differences between the computed and observed parallaxes do not exceed the probable error of the latter. Comparison of the computed parallaxes with the ones observed at different stations indicates some puzzling systematic differences between the various observatories, which need further investigation. It is curious that in general the different trigo-

nometric determinations of a parallax for which the computed value does not seem to hold different *inter se* more than for the stars for which the computed value comes out right. In the majority of these cases the computed parallax is far too large, indicating, aside from eventualities in the trigonometric parallaxes, values of $\dot{Z}$ that are smaller than -22.

I have also computed parallaxes for the stars brighter than magnitude 5 contained in Hertzsprung's Catalogue of colors[1] and in addition for all the ones south of the equation. The latter group permits a comparison with the parallaxes observed at the Southern Yale Observatory in Johannesburg. Not excluding now any stars for reason of small proper motion I find that 50% of the differences between the computed and the observed parallaxes are less than $\pm 0''.012$, the precise fraction being 60% for the stars of early types and 43% for the late types.

I have further tested the computed parallaxes by comparing the absolute magnitudes derived from them with the colors. There is a linear relation between these absolute magnitudes and Becker's[2] colors for the stars of color-indices less than $+0.100$. This relation holds even for the early B-type stars. An equally interesting correlation between color and absolute magnitude for the giant K stars corroborates the results obtained by Hall.[3] It is now possible to gain considerable insight in the nature of the giant part of the Hertzsprung diagram which cannot be obtained from spectroscopic absolute magnitudes.

The coefficient of expansion does not affect the average computed parallaxes much and fair values for the majority of the near-by stars could be obtained by simply assuming that $\dot{Z}$ has only the values 0 and -22. From their known small velocity dispersion we would expect that the lucid B stars belong with a very few exceptions also to these two groups. The components at right angles to the Z direction show dispersions increasing with the advance of the spectral type of the stars.

The hypothesis of an expansion of the stellar system incorporated in the above formula should and can easily be tested further. The expressions for the components of the proper motion and for the radial velocity as functions of the spherical coördinates on the basis of the present theory are very much like the ones for Lindblad-Oort's theory of a differential galactic rotation.[4] Except for the existence of a constant term in the new theory the equations are in fact identical, although their interpretation is different. Both theories require that the radial velocity has a gradient which is a function of the spherical coördinates, but whereas in the theory of the differential galactic rotation this gradient attains equally positive and negative values, it is never negative in the present theory. To test the relative merits of the two theories I have plotted for separate regions all published radial velocities as ordinates against the distances of the stars as abscissae. The regions are limited by the circles of galactic latitude $+20°$ and $-20°$,

the galactic equator and the circles of longitudes 0°, 22½°, 45°, etc. The distances have been computed simply from the magnitudes and the spectral types according to the table given by Kohlschütter.[5] Opposite regions have been combined into single diagrams, the distances being counted positive in one region and negative in the opposite one. The signs of the radial velocities have been reversed for negative abscissae. The slope shown by these scatter-diagrams represents the derivative of the velocity with respect to the distance. Although the scattering is large a definite slope can be recognized in most of these diagrams, and it is evident at once that the derivative is never negative. For the present I have determined the slope by simply drawing free-hand straight lines which to the average observer appear satisfactory. It should be remembered that the uncertainty in the distances may systematically influence the absolute value of the derivatives thus determined, but not their sign. If the theory of the galactic rotation were correct the derivative would increase from zero to a positive value and decrease to zero again in the direction 90° from the initial direction; it would then decrease and attain a negative value in the next octant and so on. The scatter diagrams which I shall publish shortly in a different place, show convincingly that the derivative runs from 0 to a maximum, which amounts to +5 km./sec. per 100 parsecs on the scale of Kohlschütter's distances, at 90° from the initial direction. In other words $\frac{dV}{dr}$ is proportional to $\cos^2(l - l_0')$ rather than to $\sin 2(l - l_0)$ as required by the theory of the galactic rotation. The derivative is largest in the scatter-diagrams for the regions of longitudes between 45° and 67½° whereas the theory of the galactic rotation would require it to be zero at longitude 55° assuming the center to be in longitude 325°.

[1] Ejnar Hertzsprung, *Annal. Sterrewacht Leiden*, Deel **14**, eerste stuk (1922).

[2] W. Becker, *Veröffentlichungen Universitäts-Sternwarte Berlin-Babelsberg*, Bd. X, Heft 3 (1933).

[3] J. S. Hall, *Astrophys. Jour.*, **79**, 145 (1934) and unpublished data.

[4] J. H. Oort, *Bull. Astronom. Inst. Netherlands*, **120**, 275 (1927).

[5] A. Kohlschütter, *Veröffentlichungen Universitäts-Sternwarte zu Bonn*, **22** (1930).

THE CORRELATION OF CYTOLOGICAL AND GENETICAL CROSSING-OVER IN ZEA MAYS. A CORROBORATION

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There has recently been some skepticism expressed (Brink and Cooper, 1935)[1] as to the value of the studies on the correlation of cytological and genetical crossing-over in maize published by Creighton and McClintock (1931)[6] because of the fewness of the data. Since the paper by Stern (1931)[9] dealing with *Drosophila* and having much more extensive data appeared at practically the same time and yielded the same conclusions, the authors felt it unnecessary to add to the ever-increasing amount of published work merely to record more evidence of the same nature without supplying anything essentially new or advancing. Therefore, confirmatory data which have accumulated since the time the joint paper mentioned above was published have not been considered for a separate publication. However, we now feel forced to add more data merely to counteract any suspicion that the evidence previously presented constituted insufficient proof. This will be done in as brief a form as possible, since a discussion of the method has been given in the paper mentioned above.

Chromosome 9 in maize is characterized by its relative size in the chromosome complement and by the 1:2 ratio in lengths of its two arms. The end of the short arm in some strains possesses a large knob while other strains have a very small knob or no knob. Evidence that the knob or knobless condition of a particular chromosome 9 is inherited with the same precision as a gene has been given in the previous paper and has been confirmed in many additional crosses. The knob, therefore, could be used as one cytological marker for this chromosome. The presence of an interchange between chromosomes 8 and 9 (Burnham, 1930,[2] 1934[3]; McClintock, 1930[7]) which broke chromosome 9 at a position on the long arm a short distance away from the spindle fibre attachment region provided the second cytological marker. That the genes *yg*, *c*, *sh*, *wx** lie in the interchanged chromosome which possesses the short arm of chromosome 9 has been shown by McClintock, 1931,[8] Creighton, 1934,[5] and Burnham 1934.[3,4] With reference to the knob and the interchange point, the order of the genes is knob-*yg-c-sh-wx*-interchange with *yg* very close to the knob (Creighton, 1934)[5] and *wx* close to the spindle fibre attachment region (Burnham, 1934[4] and unpublished). The standard crossover values for these genes alone are *yg-c* 21%, *c-sh* 3.3%, *sh-wx* 21%. The crossover value of *wx* to the interchange is 13.7% (Burnham, 1934[3]). That there is very little crossing-over between the knob and *yg* can be seen from the data given below.

A plant with the constitution knob-*Yg-C-Sh-Wx*-interchange was crossed to a plant with the constitution knobless-*yg-c-sh-wx*-normal. The F_1 was backcrossed to knobless-*yg-c-sh-wx*-normal. Two hundred and sixty-one individuals resulting from this backcross were examined cytologically to determine the presence or absence of the knob (knob or knobless in table below) and the presence or absence of the interchange (interchange or normal in table below) in the chromosome carrying these genes contributed by the F_1 parent. Since there are five regions in which a crossover can be detected, the results have been tabulated according to crossovers which occurred in each of these regions. The tabulated results do not represent the total backcross progeny. A higher percentage of *Yg-C-Sh-Wx* and *yg-c-sh-wx* plants were examined cytologically in an effort to obtain crossovers between the knob and *yg*. Likewise, more *Yg* plants were examined cytologically than *yg*, since plants homozygous for *yg* are reduced in vigor and often do not afford sufficient material for cytological examination.

TABLE 1

$$\frac{\text{KNOB-}Yg\text{-}C\text{-}Sh\text{-}Wx\text{-INTERCHANGE}}{\text{KNOBLESS-}yg\text{-}c\text{-}sh\text{-}wx\text{-NORMAL}} \times \text{KNOBLESS-}yg\text{-}c\text{-}sh\text{-}wx\text{-NORMAL}$$

	NUMBER OF INDIVIDUALS
Non-crossovers	
1. Knob-*Yg-C-Sh-Wx*-interchange	84
2. Knobless-*yg-c-sh-wx*-normal	45
Crossovers in region 1	
3. Knob-*yg-c-sh-wx*-normal	3
4. Knobless-*Yg-C-Sh-Wx*-interchange	1
Crossovers in region 2	
5. Knob-*Yg-c-sh-wx*-normal	13
6. Knobless-*yg-C-Sh-Wx*-interchange	11
Crossovers in region 3	
7. Knob-*Yg-C-sh-wx*-normal	3
8. Knobless-*yg-c-Sh-Wx*-interchange	3
Crossovers in region 4	
9. Knob-*Yg-C-Sh-wx*-normal	53
10. Knobless-*yg-c-sh-Wx*-interchange	18
Crossovers in region 5	
11. Knob-*Yg-C-Sh-Wx*-normal	16
12. Knobless-*yg-c-sh-wx*-interchange	3
Double crossover involving regions 2 and 4	
13. Knobless-*yg-C-Sh-wx*-normal	1
Double crossovers involving regions 4 and 5	
14. Knob-*Yg-C-Sh-wx*-interchange	5
15. Knobless-*yg-c-sh-Wx*-normal	2

It is obvious from the data given above that a genetic crossing-over between the genes *Yg-C-Sh-Wx* involves a cytological crossover between the

knob and the interchange point. These data, therefore, supplement those given in our previous publication and indicate the soundness of the conclusions drawn.

* The genes referred to in this paper by symbols are: *yg*, yellow-green plants; *c*, colored aleurone; *sh*, shrunken endosperm; *wx*, waxy endosperm.

[1] Brink, R. A., and Cooper, D. C., *Genetics*, **20**, 22–35 (1935).
[2] Burnham, C. R., these PROCEEDINGS, **16**, 269–277 (1930).
[3] Burnham, C. R., *Genetics*, **19**, 430–447 (1934).
[4] Burnham, C. R., *Am. Nat.*, **68**, 81–82 (1934).
[5] Creighton, H. B., these PROCEEDINGS, **20**, 111–115 (1934).
[6] Creighton, H. B., and McClintock, B., *Ibid.*, **17**, 492–497 (1931).
[7] McClintock, B., *Ibid.*, **16**, 791–796 (1930).
[8] McClintock, B., *Ibid.*, **17**, 485–491 (1931).
[9] Stern, C., *Biol. Zbl.*, **51**, 547–587 (1931).

TWO HEREDITARY TYPES OF HYDROCEPHALUS IN THE HOUSE MOUSE (MUS MUSCULUS)

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Different hereditary genes that independently produce a similar phenotypic effect are very common in plants but have been found only rarely in mammals. Castle and Nachtsheim (1933) have described three forms of short-hair (rex) in the rabbit, all of which look alike but depend upon different genes for their expression. They have shown that two of these genes are linked or borne in the same chromosome.

In the house mouse (Mus musculus), the gene causing blue dilution is very similar in its phenotypic effect to another that dilutes the coat in much the same manner. This second dilution character was described by J. M. Murray (1931) and called leaden. Likewise, a new pink-eye dilution in the house mouse described by Roberts (1931) resembles ordinary pink-eye dilution in its effect on eye and coat color.

The purpose of this paper is to report the results of a test designed to show whether or not two types of hydrocephalus ("water on the brain") which occur in the house mouse are due to the same gene or to different genes. One of these forms of hydrocephalus was first noticed in a flexed-tail strain of mice at Michigan State College. It proved to be a simple Mendelian recessive character (Clark, 1932) and an anatomical study (Clark, 1934) showed that it was a hydrocephalus of the obstructive or

internal type caused by closure of the brain-tube in the mid-brain. The second form of hydrocephalus was described by Dr. Klaus Zimmermann (1933) of the Kaiser Wilhelm Institute for Brain Research in Berlin, Germany. It appeared in the descendants of wild caught house mice and proved to be a simple recessive character. Plates figured by him indicate that this type of hydrocephalus is also internal in nature but he did not determine the point of obstruction in the passageways of the brain.

Affected mice of both the German and American hydrocephalic strains are very similar in appearance and are distinguished from their normal sibs by dome-shaped skulls caused by pressure of the accumulating cerebrospinal fluid trapped within the ventricles of the fore-brain. As pointed out by Zimmermann (1933) hydrocephalics of his strain appear to be somewhat dwarfed in size but the few observations I have made indicate that they soon catch up with their normal sibs in growth. Hydrocephalic mice of my strain are usually normal in size.

In testing these two types of hydrocephalus with one another, homozygous animals could not be used. According to Zimmermann (1932), affected mice of his strain may live to maturity but are invariably sterile. Hydrocephalics of the American strain in some cases live to breeding age and are fertile, but usually transmit a low-grade of hydrocephalus which is hard to detect in the offspring. Heterozygous animals of the two strains which had been tested and shown to transmit high-grade hydrocephalus were mated together. Since mice which show an extreme hydrocephalic condition at birth rarely die before one month of age, there was very little danger of such animals escaping notice, for all litters from such matings were carefully examined at birth and at frequent intervals thereafter until at least 30 days of age.

On the assumption that the same gene produces both these types of hydrocephalus, one would expect to obtain 3 normal to 1 hydrocephalic offspring from matings between heterozygotes of the two strains. Actually 65 normal mice but no hydrocephalics were obtained, a deviation of 16.25 ± 2.35 from the expected 3:1 ratio. This deviation is 6.9 times its probable error and would occur by chance alone, only once in 250,000 trials if the two types of hydrocephalus were the same. On the other hand, if we assume that the anomalies in these two strains are genetically different, matings of this type should produce only normal offspring, a result which has been realized since all of the 65 young produced were normal. It seems reasonable to conclude, therefore, that these two types of hydrocephalus depend upon different genes for their expression. This result is not surprising in view of the different early growth rates of hydrocephalic mice of the two strains.

Castle, W. E., and Nachtsheim, Hans, "Linkage Interrelations of Three Genes for Rex (Short) Coat in the Rabbit," *Proc. Nat. Acad. Sci.*, **19**, 1006–1011 (1933).

Clark, F. H., "Hydrocephalus, a Hereditary Character in the House Mouse," *Proc. Nat. Acad. Sci.*, **18**, 654–656 (1932).

—, "Anatomical Basis of Hereditary Hydrocephalus in the House Mouse," *Anat. Record*, **58**, 225–233 (1934).

Murray, Joseph M., "A Preliminary Note on the Occurrence of a Color Mutation in the House Mouse (Mus musculus)," *Science*, **73**, 1896; 482 (1931).

Roberts, E., "A New Mutation in the House Mouse (Mus musculus)," *Science*, **74**, 569 (1931).

Zimmermann, Klaus, "Eine neue Mut ationder Hausmaus: hydrocephalus," *Zeit. induktive Abstammungs- und Vererbungslehre* **64**, 176–180 (1933).

STUDIES ON THE "DIFFUSION EFFECT" UPON IONIC DISTRIBUTION

I. SOME THEORETICAL CONSIDERATIONS

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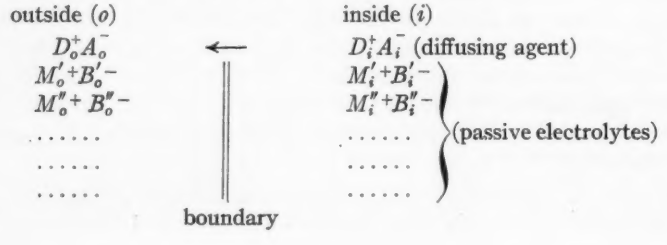
I. These studies are intended to throw some light upon certain possibilities of producing ionic distribution of the type which can occur in biological systems, i.e., great concentration differences across a boundary *permeable* to the ions under consideration.

It will be shown theoretically and in experiments that such ionic distributions can be produced in aqueous systems with at least one electrolyte *diffusing* across a boundary. It is noticeable that the theory for such systems predicts a distribution of ions of the same kind as is generally regarded as characteristic for the well-known Gibbs-Donnan equilibrium, which applies to cases where indiffusible ions are present.

Under certain conditions great "accumulations" or the contrary may be expected as a consequence of the theory given. Such conditions might well occur in biology.

II. *Assumptions.*—It seems necessary to state these in some detail.

(*a*) We consider a system of ideally dilute solutions in water of binary univalent strong electrolytes, at the same temperature.



(The arrow indicates a steady outward diffusion of DA which is kept constant.)

The boundary is in general something supporting concentration gradients. Here it may be assumed to be a membrane of a type in which water is present (the electrolytes in the membrane being strong). We will, below, refer to the boundary as the diffusion layer or the membrane (cf. footnote 11). Further it is assumed that the solutions are kept homogeneous on both sides up to the boundary by some form of convection.

(*b*) The membrane is permeable for *all* ions, but the movement of water is prevented in some way.[13] The ionic mobilities *within the membrane* (assumed constant) may be denoted by u_D, u_M, etc., for the cations and v_A, v_B, etc., for the anions.

(*c*) The ionic concentrations D_o, M'_o, M''_o, etc., and A_o, B'_o, B''_o, etc., outside are kept constant either by assuming that the volume of (*o*) is ∞ or in some other manner. The volume of (*i*) is finite.

(*d*) Also D_i^+ is *maintained constant* and the ratio $D_i^+/D_o^+ > 1$, for instance by a continuous addition of DA inside. Accordingly, *DA is a steadily diffusing electrolyte.* The following discussions are limited to the presence of only one such species. The electrolytes MB are called *passive* and may be present in any number.

(*e*) No chemical reaction takes place.[13]

(*f*) No other electric field is present besides that due to diffusion potential.

(*g*) Heat effects are ignored as well as special membrane effects.

III. *The Development of a Steady State.*—Assume that in the initial state the concentrations are $(MB)_i = (MB)_o$ and $(DA)_i > (DA)_o$ (Fig. 1). If the mobilities u_D and v_A are unequal an electrical diffusion potential, π, will arise across the membrane. Assume here that u_D is $> v_A$, then the positive current will tend to pass through the membrane from (*i*) to (*o*) and the sign of (*o*) in a fictitious external circuit will be positive. As will be shown under IV below each ion in the membrane is caused to migrate by the influence of two forces, (*a*) one due to the *concentration gradient*, and (*b*) one being an *electrical potential* gradient.

As the concentration gradients for M and B in the initial state are zero, the only force acting is the electrical one. This will move the positive M ions in the pos. $\longrightarrow$ neg. direction, i.e., from (*o*) to (*i*), and the negative B ions the opposite way. This flux for any passive ion will continue as long as the force due to the concentration gradient is less than the electrical force. Finally, however, these forces must balance each other and a steady state will be approached. The electroneutrality which would be disturbed by the separation of the ions of MB, is maintained by a compensating separation of the ions of DA. The steady state will thus be characterized by (cf. Fig. 1)

$$\left.\begin{matrix} M_i > M_o \\ B_i < B_o \\ (\text{if } u_D > v_A) \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} M_i < M_o \\ B_i > B_o \\ (\text{if } u_D < v_A) \end{matrix}\right.$$

Accordingly *a diffusion of one electrolyte may produce an accumulation or impoverishment* inside of *other* cations or anions depending upon the mobilities of the ions of the diffusing agent. This effect upon the ionic distribution arising from diffusion may be called the *diffusion effect* (cf. Teorell[1,2]).

The diffusion effect may be *permanent* or *temporary* according as the diffusion is steady or temporary. If initially, for instance, the concentra-

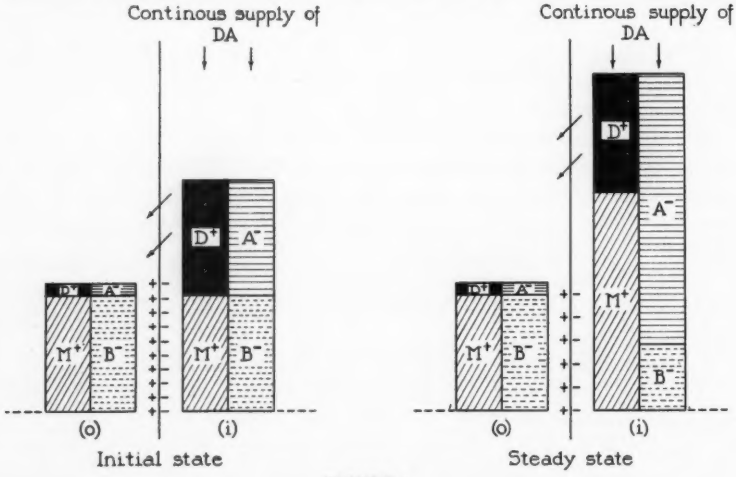


FIGURE 1

The effect of a steady diffusion of DA upon the ionic distribution when $u_D > v_A$. *DA* is by some means (by addition or production) steadily supplied "inside" (*i*). The concentrations of the electrolytes "outside" (*o*) are kept constant. An electrical potential arises across the boundary (membrane). This causes an inward migration of M^+ and an outward migration of B^-. Finally the concentration gradients have become sufficiently large to balance the electrical gradient and the system approaches a steady state. Thus the *M* ions became "accumulated" and the *B* ions diminished in amount inside.

tion $(DA)_i > (DA)_o$, but no additional supply of $(DA)_i$ is furnished, $(DA)_i$ will diffuse out and approach the value of $(DA)_o$. During this diffusion, however, a temporary diffusion effect will appear.[4] This has been experimentally demonstrated by the author[1,2] and by Norberg.[3]

A diffusion effect arising from a permanent diffusion has qualitatively been shown in some experiments by Straub,[5] who spoke of a "Harmonie-Einstellung" of the ions.[6] A series of experiments including potential

measurements performed by the author confirm the theory of the diffusion effect also from a quantitative point of view. These results will be dealt with in another paper.

IV. (*a*) *A quantitative treatment* of the influence of the diffusion effect on the distributions of interest, namely, M_i'/M_o', M_i''/M_o'', etc., and B_i'/B_o', etc., *valid for the steady state*, can be performed in a classical way if we follow the kinetic treatment of electrolyte diffusion first given by Nernst[7] and more rigorously developed by Planck.[8]

The migration of any ion species through a surface element in the membrane of the thickness dx, where its concentration is C, is caused by superposition of two forces:

(1) one due to a *concentration gradient*, this force is per gram ion equal to $RT \cdot \frac{1}{C}\frac{dC}{dx}$.

(2) one due to the *electrical potential gradient*, written $F \cdot \frac{d\varphi}{dx}$.

Here R, the gas constant, is expressed in joules. C is the concentration in gram ion/liter. T is the absolute temperature. F is the charge of 1 gram equivalent in coulombs and φ the electrical potential in volts. The number of moles of any ion species which passes through the surface element in the unit time, $\frac{dC}{dt}$, or the flux, ϕC, is equal to *mobility* $\times$ *concentration* $\times$ *force per gram ion* or in general (u denotes mobility),

$$\phi C = u \cdot C \cdot \left(\frac{RT}{C}\frac{dC}{dx} \mp F\frac{d\varphi}{dx} \right). \tag{1}$$

The sign before the electrical term is taken negative when cations are concerned and positive for anions. "ϕC" means "the flux of C" (not a product).

In the *steady state* developed by a permanent diffusion[9] there is no flux of the passive ion species M^+ and B^-:

$$\phi M' = \phi M'' \text{ etc.} = \phi B' = \phi B'' = 0 \tag{2}$$

and

$$\phi(\Sigma M) = \phi(\Sigma B) = 0. \tag{3}$$

Using equation (1) for the conditions (2) gives for any passive cation species the flux, ϕM, as

$$u_M \cdot M\left(\frac{RT}{M}\frac{dM}{dx} - F\frac{d\varphi}{dx} \right) = 0 \tag{4}$$

and for any passive anion species the flux, ϕB, as

$$v_B \cdot B\left(\frac{RT}{B}\frac{dB}{dx} + F\frac{d\varphi}{dx} \right) = 0 \tag{5}$$

which leads to

$$\frac{d\varphi}{dx} = \frac{RT}{F} \cdot \frac{1}{M} \frac{dM}{dx} = -\frac{RT}{F} \frac{1}{B} \frac{dB}{dx}. \qquad (6.1 \text{ and } 6.2)$$

Integration between the limits (o) and (i) gives the diffusion potential π:

$$\pi = \varphi_o - \varphi_i = \frac{RT}{F} \int_i^o \frac{1}{M} \frac{dM}{dx} dx \qquad (7)$$

or

$$\pi = \frac{\mathbf{RT}}{\mathbf{F}} \ln \frac{\mathbf{M}_i}{\mathbf{M}_o} = \frac{\mathbf{RT}}{\mathbf{F}} \ln \frac{\mathbf{B}_o}{\mathbf{B}_i}. \qquad (8.1 \text{ and } 8.2)$$

It now becomes obvious that *the steady state* of a system of this nature with a steady diffusion *is characterized by a constant ratio series*

$$\frac{M_i'}{M_o'} = \frac{M_i''}{M_o''} = \ldots = \frac{\Sigma M_i}{\Sigma M_o} = \frac{\Sigma B_o}{\Sigma B_i} = \ldots = \frac{B_o''}{B_i''} = \frac{B_o'}{B_i'} = \xi. \qquad (9)$$

A distribution of this type (9) has been hitherto generally regarded as typical for the so-called Donnan equilibrium. As a matter of fact it will be shown below (VII) that the genuine Donnan equilibrium appears to be a limiting case of the more general case discussed here, where we are dealing with *differences* in permeability instead of with *im*permeability.

(b) The formula (8.1 and 8.2) may be written

$$\pi F = RT \ln \frac{M_i}{M_o} = RT \ln \frac{B_o}{B_i}. \qquad (10)$$

The work relation (10) may be interpreted as follows: *the unequal ionic distribution is produced and maintained by the presence of a diffusion membrane potential.* This arises from a diffusion of what may be called an "active" or "diffusing agent."

(c) As the diffusion potential π is equal to the expression for the electrode potential for any passive ion, a concentration chain consisting of reversible electrodes for such an ion species placed in the solution (o) and (i) should give no current, i.e., the energy obtained = 0. This conclusion may be expressed thus: when the system as a whole is in a steady state *the passive ionic constituents are in a true state of equilibrium,* just as in the Donnan equilibrium. "Equilibrium" is here interpreted as a state of matter from which no free energy can be obtained.

V. *The magnitude of the distribution ratio, in the steady state,* $\frac{M_i'}{M_o'} = \frac{M_i''}{M_o''} = \ldots = \frac{B_o'}{B_i'} = \frac{B_o''}{B_i''} = \ldots = \xi$ (9), can be calculated in several somewhat different ways.

(*a*) If the diffusion potential π is regarded as known then equation (8) could be used directly. (*b*) π can, however, be computed from a knowledge of the ionic composition and ionic mobilities. Planck[8] and Henderson[10] have proposed somewhat different formulas. As the values of M_o and B_o are assumed to be fixed, the unknown M_i or B_i must be computed so they satisfy simultaneously in equation (8) and in either of the formulas just referred to.[11] Planck suggests for the calculation of π the equation $\pi = \frac{RT}{F} \ln \xi$, where ξ is defined by a transcendental expression (loc. cit. or text books). A comparison of Planck's $\pi = \frac{RT}{F} \ln \xi$, with equation (8.1) $\pi = \frac{RT}{F} \ln \frac{M_i}{M_o}$ shows that Planck's ξ is identical with our desired ratio M_i/M_o, etc. Planck's formulas could accordingly be used and as a matter of fact a combination of Planck's transcendental expression for ξ with equation (8) gives the same result as will be obtained below (equation 18). The Henderson formula gives formally a different result but numerically values in close agreement with the results obtained here.

(*c*) We will, however, show that an expression for these distribution ratios, now called ξ, can be derived in consistency with the treatment already used under IV.

The *steady state* must, besides the zero flux condition (equations 2, 3) of the passive ions, in regard to the diffusing agent be characterized by

$$\phi D^+ = \phi A^-. \tag{11}$$

Proceeding as under IV and substituting in (11) we get

$$u_D \cdot D\left(\frac{RT}{D}\frac{dD}{dx} - F\frac{d\varphi}{dx}\right) = v_A \cdot A\left(\frac{RT}{A}\frac{dA}{dx} + F\frac{d\varphi}{dx}\right) \tag{12}$$

which solved for $\frac{d\varphi}{dx}$ gives

$$\frac{d\varphi}{dx} = \frac{RT}{F} \cdot \frac{u_D \cdot dD - v_A \cdot dA}{u_D \cdot D + v_A \cdot A} \cdot \frac{1}{dx}. \tag{13}$$

We now sum the equations of type (6) for all M (or B) species and write $\frac{d\varphi}{dx} = \frac{RT}{F} \cdot \frac{1}{(\Sigma M)} \frac{d(\Sigma M)}{dx}$ and combine this with (13). Further we substitute the A terms, making use of the electroneutrality condition $D + \Sigma M = A + \Sigma B$ and obtain for (13)

$$\frac{RT}{F} \cdot \frac{1}{(\Sigma M)} \frac{d(\Sigma M)}{dx} = \frac{RT}{F} \cdot \frac{u_D \cdot dD - v_A[dD + d(\Sigma M) - d(\Sigma B)]}{u_D \cdot D + v_A\,[D + (\Sigma M) - (\Sigma B)]} \cdot \frac{1}{dx}. \tag{14}$$

$d(\Sigma B)$ is substituted by $-\left(\frac{(\Sigma B)}{(\Sigma M)}\right) d(\Sigma M)$, a relation directly obtained from equations of type (6). Equation (14) is then solved for $\frac{dD}{d(\Sigma M)}$ when the (ΣB) terms will cancel out. The result is

$$\frac{dD}{d(\Sigma M)} = a \frac{D}{(\Sigma M)} + b \tag{15}$$

where the constants a and b denote $\frac{u_D + v_A}{u_D - v_A}$ and $\frac{2v_A}{u_D - v_A}$, respectively. The differential equation (15) gives a solution

$$D - \frac{b}{1 - a}(\Sigma M) - I \cdot (\Sigma M)^a = 0 \tag{16.1}$$

or

$$D + (\Sigma M) - I \cdot (\Sigma M)^a = 0 \tag{16.2}$$

the I being an integration constant. This is eliminated by dividing two equations (16) applied to the limits (i) and (o):

$$\frac{D_i + \Sigma M_i}{D_o + \Sigma M_o} = \frac{I \cdot (\Sigma M_i)^a}{I \cdot (\Sigma M_o)^a}. \tag{17}$$

Finally taking the logarithms, substituting a, and considering the relations of equation (9) we obtain

$$\log \xi = \log \frac{\mathbf{M}_i'}{M_o'} = \ldots = \log \frac{B_o'}{\mathbf{B}_i'} = \ldots = \log \frac{(\Sigma \mathbf{M}_i)}{(\Sigma M_o)} = \frac{u_D - v_A}{u_D + v_A} \log \frac{D_i + (\Sigma \mathbf{M}_i)}{D_o + (\Sigma M_o)}. \tag{18.1}$$

It should be noticed that the terms $D + (\Sigma M)$ are equal to the "total" concentration C. For a single ion species, for instance M', we may write

$$\log \frac{\mathbf{M}_i'}{M_o'} = \frac{u_D - v_A}{u_D + v_A} \log \frac{D_i + \frac{(\Sigma M_o)}{M_o'} \mathbf{M}_i'}{D_o + (\Sigma M_o)}. \tag{18.2}$$

The transcendental expression (18.1.2) contains the desired quantitative description of the diffusion effect *valid for the steady state.* The terms which according to the assumption made under II are considered as known are printed in common type, those which are variables in heavy type.

Numerical solution of equation (18) can be performed either (a) by a

graphical computation, the intersection of the curves for the left, respectively right-hand member obtained by inserting different values of the variables will be the solution, or (*b*) by a method of successive approximation.

VI. *The factors influencing the magnitude of the distribution ratio* ξ (*in the steady state*) can be read off from equation (18). They are the ratios u_D/v_A, $D_i/\Sigma M_o$ and D_i/D_o. *The larger any of these ratios are the greater the "accumulation" of* M^+ *ions* "inside" (and correspondingly the decrease of B^- ions), provided $u_D > v_A$. If $u_D < v_A$ it is the B^- ions which will accumulate, etc.

To get an impression of the very marked concentration differences which theoretically can be expected under certain conditions the tables 1 and 2 should be consulted.

TABLE 1

THE VALUE OF THE DISTRIBUTION RATIO $\xi \left(= \frac{M_i'}{M_o'}, \text{ ETC.} \right)$

(u_D/v_A is assumed to be 5)

$D_i/\Sigma M_o$	D_i/D_o = 1	10	100	1000	10,000	∞ ($D_o = 0$)
0.01	1	1.02	1.02	1.02	1.02	1.02
0.1	1	1.12	1.17	1.18	1.18	1.18
1	1	1.86	2.12	2.14	2.15	2.15
10	1	3.58	6.0	6.4	6.4	6.49
100	1	4.49	15.0	23.5	25.0	25.2
1000	1	4.64	20.5	56	101	107
10000	1	4.65	21.4	94	298	479
∞	1	4.65	21.5	100	465	∞

TABLE 2

THE DISTRIBUTION RATIO $\xi \left(= \frac{M_i'}{M_o'}, \text{ ETC.} \right)$ AT DIFFERENT MOBILITY RELATIONS

(It is assumed that $D_i = \Sigma M_o$)

D_i/D_o	MOBILITY RATIO u_D/v_A = 1	2	5	10	100	1000	DONNAN EQUILIBRIUM ($v_A = 0$) ∞
∞	1	1.33	2.15	3.31	16.9	>100	∞
100	1	1.32	2.12	3.13	15.0	85	100
10	1	1.27	1.86	2.70	6.9	9.9	10
1	1	1	1	1	1	1	1

VII. *The Donnan equilibrium* can, as pointed out above, be regarded as a limiting case of the case discussed in this paper. If one of the ions of *DA* is prevented from diffusing, i.e., making either v_A or u_D equal to zero and the addition or production of *DA* is stopped, the equation (18.1) is transformed into

i. $(v_A = 0)$ $$\log \frac{\Sigma M_i}{\Sigma M_o} = \log \frac{D_i + \Sigma M_i}{D_o + \Sigma M_o} \tag{15.1}$$

which means (remembering equation (9)) that

$$\frac{M_i'}{M_o'} = \frac{M_i''}{M_o''} = \ldots = \frac{D_i}{D_o} = \ldots = \frac{B_o''}{B_i''} = \frac{B_o'}{B_i'} \tag{15.2}$$

ii. $(u_D = 0)$ $$\log \frac{\Sigma M_i}{\Sigma M_o} = -\log \frac{D_i + \Sigma M_i}{D_o + \Sigma M_o} = \log \frac{C_o}{C_i} = \log \frac{A_o + \Sigma B_o}{A_i + \Sigma B_i} \tag{16.1}$$

which must be interpreted

$$\frac{M_i'}{M_o'} = \frac{M_i''}{M_o''} = \ldots = \frac{A_o}{A_i} = \ldots = \frac{B_o''}{B_i''} = \frac{B_o'}{B_i'}. \tag{16.2}$$

Equations (15.2) and (16.2) are the equilibrium conditions requested by Donnan. The approach of the diffusion effect to the Donnan effect with increasing u_D/v_A (v_A tends to 0) is evident from table 2.

VIII. *The validity of the conclusions* above depends on the validity of the assumptions given under II. In experimental work with solutions which are not ideal one might expect deviations due to differences in activity, water migration, special influence of the membrane and several other complicating factors.[12] Experiments by the author show that this is the case. In dilute solutions, however, the agreement between theory and experiments can be considered good. These results and the discussion of the bearings of these theories on certain biological ionic distributions will be published in subsequent papers.

I wish to express my acknowledgment to Drs. W. J. V. Osterhout and L. G. Longsworth for valuable criticism.

Summary.—1. The behavior of an aqueous system of strong electrolytes is studied, where a steady diffusion across an ion-permeable boundary (membrane) takes place.

2. Theoretical considerations predict, for the assumptions given, (*a*) the possibility of the development of concentration differences in respect to "passive" ions ("diffusion effect" upon ionic distribution); (*b*) an approach to a steady state of the system as a whole, where the "passive" ions are in a true state of equilibrium characterized by the relations

$$\frac{\text{(conc. of cation) inside}}{\text{(conc. of cation) outside}} = \frac{\text{(conc. of anion) outside}}{\text{(conc. of anion) inside}} = \xi.$$

3. The genuine Donnan equilibrium appears as a limiting case of the more general case treated here. The "*diffusion effect*" appears when a concentration gradient is present and is due to *differences* in ionic mobility

("*relative* impermeability"), whereas the "*Donnan effect*" is caused by "*absolute* impermeability" of one or several ion species.

4. A formula for the distribution ratio in the steady state has been derived and the factors which have influence upon it are pointed out. Some tables are given showing the calculated distribution ratio for various assumptions.

5. Theoretically very marked concentration differences across a boundary (membrane) are possible. It is emphasized, however, that deviations from the theory might occur, because the validity of the assumptions may be limited and factors not involved in the theory might play a rôle.

6. Diffusion effects of the type presented in this paper may be of biological importance.

* Rockefeller Foundation Fellow.

[1] Teorell, T., *J. Physiol.*, **78**, 11P (1933).

[2] Teorell, T., *Skand. Arch. Physiol.*, **66**, 225 (1933).

[3] Norberg, B., *Pflüger's Arch.*, **234**, 200 (1933).

[4] The possibility of a temporary migration of one ion species, initially uniformly distributed, in a system with one electrolyte diffusing was mentioned shortly by Abegg and Bose (Abegg, R., and Bose, E., *Z. physik. Chem.*, **30**, 545 (1899)) in a footnote (p. 549). Cf. recent theory of interdiffusion by Taylor (Taylor, P. B., *J. Phys. Chem.*, **31**, 1478 (1927).

[5] Straub, J., *Kolloid-Z.*, **62**, 13 (1933). Straub (*Ibid.*, **64**, 72 (1933)) connects his results in an interesting way with certain conceptions introduced by Schreinemakers.

[6] Some related effects during dialysis are recently reported by Brintzinger and Wallach (Brintzinger, H., and Wallach, J., *Kolloid-Z.*, **68**, 36 (1934)).

[7] Nernst, W., *Z. physik. Chem.*, **2**, 617 (1888); **4**, 154 (1889).

[8] Planck, M., *Wied. Ann.*, **40**, 561 (1890).

[9] A treatment similar to this given here can easily be applied to a temporary diffusion effect at the maximum condition.

[10] Henderson, P., *Z. physik. Chem.*, **59**, 118 (1907).

[11] Planck and Henderson (loc. cit.) have different assumptions concerning the conditions in the diffusion layer. They do not enter into our discussion. Our treatment, however, is in principle that of Planck and experiments (unpublished) on the ionic distribution *within* the diffusion layer confirm Planck's assumptions rather than those of Henderson.

[12] McBain and Liu (McBain, J. W., and Liu, T. H., *J. Am. Chem. Soc.*, **53**, 59 (1931)) and McBain and Dawson (McBain, J. W., and Dawson, C. R., *Ibid.*, **56**, 52 (1934)) assume a "collision effect" in more concentrated solutions (loc. cit.).

[13] Extensions of the theory given here, involving the influence of water migration and of chemical reactions, will be presented in subsequent communications.

ON THE FOUNDATIONS OF A CONSTRUCTIVE THEORY OF DISCRETE COMMUTATIVE ALGEBRA (SECOND PAPER)

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In the first paper[1] under this title I mentioned certain topics which may be treated from a constructive standpoint in commutative algebra, and discussed some of these topics in detail. In the present paper some of the topics mentioned but not discussed in detail in the first paper will be considered further.

1. To lead up naturally to extension of the ring concept, we first define a ring as a system of elements which form an Abelian group under addition, a semi-group under multiplication and the usual right- and left-hand distributive laws hold. This form of definition immediately suggests the generalization obtained by replacing the words "Abelian group" by "semi-group" in the above. If this is done we shall call the resulting system a *semi-ring* or *associative algebra*. The finite arithmetic in which the cancellation law of addition does not hold, which was referred to in §1 of our first paper, is, in general, a semi-ring but not a ring. Other examples may be found in number theory.

A ring in which division is uniquely possible except when the divisor is a zero divisor will be termed a *quasi-field*. It is easily shown that any finite ring is a quasi-field, and the ring of square matrices of order n with elements in a field has this property. It seems to me that it is precisely the quasi-field which characterizes nearly all the systems which have so far been studied under the name *algebra*, whereas the term *number-theory* is generally applied to the study of rings and semi-rings containing only what we are pleased to call integral elements. It is through the study of rings in general and perhaps also semi-rings that we may expect a unification of the two subjects.

As to the extent of the theory which may be developed by constructive methods a large part of the theory of algebraic numbers may be included such as the theories of the additive basis for integers, the units and the discriminant. However, so far the determination of the *class-number* of an algebraic field has not yet been effected without the use of analysis.

The Galois theory may be developed in a constructive way much as usual. We may, however, avoid the use of the concept of polynomials with indeterminates as roots by determining particular polynomials of degree n with rational coefficients whose group is the symmetric group[2]

in connection with the proof that there exist equations of degree > 4 which are not solvable algebraically.

2. As to the extension of field theory to rings we proceed a bit differently from Krull[3] and it is found convenient, in order to obtain natural analogies with theories concerning fields, to make use of a certain modification of the concept of equality. Two elements of a ring R are said to be absolutely distinct if their difference is not a zero-divisor, but if the difference is a zero-divisor, the elements are said to be semi-equal. For example, in a ring, the solutions of $ax = b$ are semi-equal, if any exist.

As to finite rings, the discussion of this topic is much simplified by the use of a certain representation of the elements of a finite Abelian group G of order p^n, p prime and generalizations of this notion. In any such group it may be shown that there exist elements $A_1, A_2, \ldots, A_{e_0}$ of G such that all the elements of G are given uniquely by

$$\prod_{c=0}^{\alpha-1} \prod_{k=0}^{e_c} A_k^{x_{ck}p^c}; \quad e_{c-1} \geqq e_c$$

where each x_{ck} ranges independently over the set $0, 1, \ldots, p-1$. This follows from the known existence of a basis of G but it may be proved independently and the existence of a basis may be shown to follow from it. In the additive form it takes on what may be called the *p-adic representation of an Abelian group of order* p^n, p prime, as follows:

$$\sum_{c=0}^{\alpha-1} \sum_{k=1}^{e_c} p^c x_{ck} A_k.$$

It has an obvious connection with Hensel's theory of p-adic numbers. If we employ group operators we may replace x_{ck} in the above by elements of any finite field and may apply the result to the theory of finite rings. There is an extension of this to infinite rings, in fact these ideas appear to afford the basis for a theory of *linear dependence in rings*.

The theory of the divisors of the discriminant of an algebraic field may be discussed by employing the theory of finite rings.

The ring methods employed enable us to extend to certain other types of finite rings the theorem of Wedderburn which states that a finite ring, in which all the elements aside from the zero element form a group under multiplication, is commutative.

3. As to the ordering of algebraic numbers we first consider a polynomial $f(x)$ with coefficients in the rational field F and irreducible in F and of degree n. Suppose we have, in the decomposition field of $f(x)$

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

Assume also that there are elements a and b of F such that

$$f(a)f(b) < 0$$

then we call (a,b), $a < b$, a *root interval* of $f(x)$. In order to build up, based

on this notion, the idea of real field, it is necessary to isolate a root interval (a_1,b_1), called a *simple interval* in (a,b) such that we cannot have root intervals (a_2,b_2) and (a_3,b_3) with $a_2 \geqq a_1$, $b_3 \leqq b_1$ and $f(b_2)f(a_3) < 0$. This may be done in a finite number of steps using the following result obtained by Kronecker[4] using only constructive methods:

If there exist two root intervals of $f(x)$ say (a_1,b_1) and (a_2,b_2) with $a_2 < b_1$ such that $f(b_2)f(a_1) > 0$ and such that if k is any rational number included in either interval then $f'(k) > c$ where c is some positive rational number and $f'(x)$ does not change sign in either interval, then (b_1,a_2) is a root interval of $f'(x)$.

To apply this to our problem we find intervals in (a,b) so small that the condition $|f'(k)| > c$ is satisfied and we then make the determination of a simple root interval of $f(x)$ depend on finding simple root intervals of $f'(x)$ and apply induction.

After a simple root interval has been so determined, we may then associate one of the roots of $f(x) = 0$ with the simple root interval (a,b) and call it α. Then α is said to be *real* and to generate a *real algebraic field*. To order the elements in $F[\alpha]$ we employ a modification of the method proposed by Schatunovsky.[5] Write an element, $\neq 0$, of $F(\alpha)$ in the form

$$d_0 + d_1\alpha + \ldots + d_{n-1}\alpha^{n-1}$$

where the d's are rational. Let

$$\varphi(x) = d_0 + d_1x + \ldots + d_{n-1}x^{n-1}.$$

Consider $\varphi(a)$ and $\varphi(b)$. If they differ in sign we may find a sub-interval of (a,b) say (a_1,b_1) which is a root interval of $f(x)$ but so that $\varphi(a_1)\,\varphi(b_1) > 0$, and we say that $\varphi(\alpha)$ is positive if $\varphi(a_1)$ is positive and negative if $\varphi(a_1)$ is negative. Since $f(x)$ is irreducible, an element of $F[\alpha]$ is either positive, negative or zero. Adopting these conventions and the symbols $>$ and $<$ we may show that if B and Y are in $F[\alpha]$ with $B > 0$ and $Y > 0$ then $B + Y > 0$ and $BY > 0$.

An extension of this method enables us to order a real algebraic field formed by the adjunction of the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ say F' so that we obtain in F'

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_k)\, f_1(x)$$

and where the coefficients of $f_1(x)$ are in F' and where $f_1(x)$ has no root intervals in F'. By the use of part of Gauss' argument in connection with his second proof of the "fundamental theorem of algebra," we are able to adjoin a finite number of real fields to F' and to obtain a real field F'' in which $f_1(x)$ decomposes into factors of degree = 2, which result takes the place in our constructive theory that the fundamental theorem just mentioned takes in the usual theory.

By extending the above methods, it is possible to order the elements of certain quasi-fields which are not fields. Here the elements are either positive, negative or zero-divisors.

[1] These PROCEEDINGS, **20**, 579–584 (1934).

[2] Van der Waerden, *Moderne Algebra*, Erster Teil, pp. 130–131.

[3] *Mathematische Annalen*, bd. 88, p. 80 and bd. 91, p. 1.

[4] H. B. Fine, *Bull. Amer. Math. Soc.* (1913).

[5] Algebra as a study of congruences with respect to functional moduli (Russian). Odessa (1913), Chap. 8. I am indebted to Dr. A. E. Ross for an English translation of this.

CONFORMAL-AFFINE CONNECTIONS

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1. R. Weitzenbock[1] has indicated how to associate with a projective connection[2] Π^i_{jk} an affine connection T^i_{jk} which is at the same time a projective invariant, i.e., the components T^i_{jk} are functions of the Π^i_{jk} and their derivatives. The method used depends on the existence of a relative projective scalar invariant. For the general space with a projective connection such scalars exist. The affine connection T^i_{jk} is of course not uniquely determined for a given space but depends on the choice of the scalar.

In this note the method used by Weitzenbock is applied to the conformal connection K^i_{jk}, and an affine connection J^i_{jk} is obtained which is a conformal invariant, i.e., it is a function of the G_{ij}[3] and their derivatives, where the G_{ij} are components of the fundamental conformal tensor of a metric space V_n. This affine connection will be called the conformal-affine connection.

A complete set of invariants for the determination of the conformal equivalence of metric spaces is also obtained, these invariants involving the curvature tensor, based on the J^i_{jk}, and its successive covariant derivatives.

2. The components K^i_{jk} of the conformal connection transform by[4]

$$\overline{K}^a_{bc}\frac{\partial x^i}{\partial \bar{x}^a} = \frac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^c} + K^i_{jk}\frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial x^k}{\partial \bar{x}^c} - \frac{1}{n}\left(\frac{\partial x^i}{\partial \bar{x}^b}\bar{\psi}_c + \frac{\partial x^i}{\partial \bar{x}^c}\bar{\psi}_b - \overline{G}_{bc}\overline{G}^{ts}\frac{\partial x^i}{\partial \bar{x}^t}\bar{\psi}_s\right) \tag{2.1}$$

under an arbitrary coördinate transformation

$$x^i = x^i(\bar{x}^1, \ldots, \bar{x}^n).$$

The $\bar{G}_{bc}$ and $\bar{\psi}_i$ are given by

$$\bar{G}_{bc} = G_{ij} \frac{\partial x^i}{\partial \bar{x}^b} \frac{\partial x^j}{\partial \bar{x}^c} \Delta^{-\frac{2}{n}}, \quad \left(\Delta = \left| \frac{\partial x}{\partial \bar{x}} \right| \right),$$

$$\bar{\psi}_i = \frac{\partial \log \Delta}{\partial \bar{x}^i}.$$

Suppose now that V_n admits a conformal relative scalar T which we can assume without loss of generality to be of weight one. When T transforms by

$$\bar{T} = \Delta T,$$

from which we derive

$$\frac{\partial \log \Delta}{\partial \bar{x}^i} = \frac{\partial \log \bar{T}}{\partial \bar{x}^i} - \frac{\partial \log T}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^i}. \tag{2.2}$$

If now we substitute from (2.2) in (2.1) we obtain

$$\bar{J}^a_{bc} \frac{\partial x^i}{\partial \bar{x}^a} = \frac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^c} + J^i_{jk} \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial x^k}{\partial \bar{x}^c}, \tag{2.3}$$

where

$$J^i_{jk} = K^i_{jk} + \frac{1}{n} \left(\frac{\partial \log T}{\partial x^j} \delta^i_k + \frac{\partial \log T}{\partial x^k} \delta^i_j - G_{jk} G^{is} \frac{\partial \log T}{\partial x^s} \right), \tag{2.4}$$

and simlar expressions for $\bar{J}^i_{jk}$ in the barred quantities of the right members of (2.4). The quantities J^i_{jk} are thus the components of an affine connection constructed from the G_{ij} and their derivatives (since T is so constructed). In this way for every relative conformal scalar T of V_n we can associate a conformal-affine connection, these connections being distinct for different choices of T. Hence we cannot have a unique conformal-affine connection associated with a V_n.

3. To obtain a complete set of conformal tensor invariants for V_n, consider the following mixed system

$$\frac{\partial x^i}{\partial \bar{x}^a} = u^i_a, \tag{3.1}$$

$$\frac{\partial u^i_b}{\partial \bar{x}^a} = \bar{J}^s_{ba} u^i_s - J^i_{jk} u^j_b u^k_a, \tag{3.2}$$

$$\bar{G}_{ab} = | u^l_m |^{-\frac{2}{n}} G_{ij} u^i_a u^j_b. \tag{3.3}$$

If (3.3) be differentiated and (3.1), (3.2) used to eliminate derivatives of x^i and u^i_j the resulting equations are satisfied identically. The equations

(3.1) and (3.2) are of the same form as those used in the equivalence problem of affine spaces.[5] Hence we obtain the sequence giving the transformation of the curvature tensor components J^i_{jkl} and its covariant derivatives $J^i_{jkl,m_1,\ldots,m_r}$ based on J^i_{jk}. This gives us the theorem.

The conformal tensors

$$G_{ij},\ J^i_{jkl},\ J^i_{jkl,m},\ \ldots,$$

form a complete set of invariants for the conformal equivalence problem for a V_n admitting a relative scalar T.

4. If $n > 3$, one of the simplest relative conformal scalars is given by

$$T_1 = (G^{ab}G^{cd}\,{}^\circ B^h_{iac}\,{}^\circ B^i_{hbd})^{-\frac{n}{4}},$$

where ${}^\circ B^i_{jkl}$ are the components of the Weyl conformal curvature tensor.[6] It is easily shown that $T_1 \not\equiv 0$. Other forms for T can be obtained from relative scalars of the type

$${}^\circ B^{a_1}_{a_2a_3a_4}\,{}^\circ B^{b_1}_{b_2b_3b_4}\ \ldots\ {}^\circ B^{r_1}_{r_2r_3r_4}\,\epsilon^{s_1\ldots s_n}\ldots\,\epsilon^{i_1\ldots i_n},$$

where the ϵ's are the components of the numerical tensor density; and there are r ${}^\circ B$'s and $k\epsilon$'s in the above product, where $2r = nk$. The indices are to be summed in a manner to eliminate all free indices. Expressions for T involving third and higher order derivatives of the G_{ij} can also be easily obtained.

When $n = 3$ the components ${}^\circ B^i_{jkl}$ vanish identically. It is possible to construct relative scalars from the components[7] ${}^\circ B^o_{ijk}$. One such, not identically zero is

$$G^{ia}G^{jb}G^{kc}\,{}^\circ B^o_{ijk}\,{}^\circ B^o_{abc}.$$

In case a V_n is conformally flat the above methods for obtaining conformal scalars fail. In fact for such spaces all conformal tensor invariants involving derivatives of the G_{ij} are identically zero. However, two such spaces are equivalent if the signatures of the quadratic forms with the G_{ij} as coefficients are the same, so that conformal tensor invariants are not needed for the equivalence problem.

It is interesting to note that the covariant derivatives of G_{ij} and T taken with respect to J^i_{jk} are identically zero.

[1] *Akad. Wetenschappen Amsterdam, Proc.*, **35**, 2, 1220–1229 (1932).

[2] Throughout, all indices have the values $1, \ldots, n$ $(n > 2)$.

[3] T. Y. Thomas, *Differential Invariants of Generalized Spaces*, Cambridge Univ. Press (1934), p. 66. We indicate this reference by *T*.

[4] *T*, p. 68.

[5] *T*, p. 204.

[6] *T*, p. 72.

[7] *T*, p. 73.

FORMALISM FOR CONFORMAL GEOMETRY

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1. Conformal geometry confines itself to the measure of angles between directions. The components D^i of a direction at a point x are given by

$$D^i = \rho dx^i$$

where ρ is an arbitrary positive number. If we change to a new coördinate system and a new value of ρ these components are transformed according to the law,

$$\bar{D}^i = V^i_j D^j \tag{1.1}$$

where

$$V^i_j = V^0_0 \frac{\partial \bar{x}^i}{\partial x^j}, \quad 0 < V^0_0 = \text{arbitrary}. \tag{1.2}$$

If g_{ij} are components in a coördinate system x of the metric tensor of a Riemannian geometry, the components in any coördinate system $\bar{x}$ of any tensor which yields the same angular measure are given by

$$\bar{g}_{ij} = g_{kl} U^k_i U^l_j \tag{1.3}$$

where

$$U^i_j = U^0_0 \frac{\partial x^i}{\partial \bar{x}^j}, \quad 0 < U^0_0 = \text{arbitrary}. \tag{1.4}$$

Differentiating (1.3) we find the following law of transformation for the Christoffel symbols of the g's:

$$\overline{\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}} = V^c_k \left(\left\{ \begin{matrix} c \\ ab \end{matrix} \right\} U^a_i + \frac{\partial U^c_i}{\partial x^b} \right) \frac{\partial x^b}{\partial \bar{x}^j} + \delta^k_j \frac{\partial \log U^0_0}{\partial \bar{x}^i} - \bar{g}^{kl} \bar{g}_{ij} \frac{\partial \log U^0_0}{\partial \bar{x}^l} \tag{1.5}$$

where now we assume

$$U^0_0 = \frac{1}{V^0_0} \text{ and hence } U^i_j V^j_k = \delta^i_k. \tag{1.6}$$

2. As in the generalized projective geometry,[1] a change of gauge may be associated with a transformation of a gauge variable,

$$\bar{x}^0 = x^0 + \log U^0_0, \tag{2.1}$$

and this may be considered simultaneously with a coördinate transformation,

$$\bar{x}^i = \bar{x}^i(x). \tag{2.2}$$

The invariant theory of the transformation (2.1) and (2.2) may be worked out in terms of projective tensors of which the law of transformation in the case of contravariant vectors is

$$\bar{T}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} T^\beta \tag{2.3}$$

where the indices run over the range, 0, 1, ..., n. In what follows we shall use small Greek letters from the early part of the alphabet for this range, reserving small Latin letters for the range, 1, 2, ..., n.

For our present purposes it seems desirable to introduce also another class of projective tensors, namely those having a law of transformation typified by

$$\bar{P}^\alpha = V^\alpha_\beta P^\beta, \tag{2.4}$$

where

$$V^\alpha_\beta = V^0_0 \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \tag{2.5}$$

or

$$V^k_0 = 0 < V^0_0 = \text{arbitrary}, \; V^0_k = \frac{\partial V^0_0}{\partial x^k}, \; V^i_j = V^0_0 \frac{\partial \bar{x}^i}{\partial x^j}. \tag{2.6}$$

We may note that the components P^i of a contravariant projective vector P^α according to (2.4) are the components of a direction according to (1.1). Also that all the components of all the metric tensors are components of a single covariant projective tensor $g_{\alpha\beta}$ subject to the invariant condition

$$g_{\alpha 0} = g_{0\alpha} = 0. \tag{2.7}$$

Furthermore, if φ_α is a covariant projective vector according to the new definition, φ_α/φ_0 is a covariant projective vector according to the old definition.

3. We now observe that if φ_α is a covariant projective vector according to the new definition, the quantity $g^{ij}\varphi_i\varphi_j$ has the law of transformation,

$$\bar{g}^{ij}\bar{\varphi}_i\bar{\varphi}_j = g^{ab}\varphi_a\varphi_b + 2\varphi_a\varphi_0\bar{g}^{ij}U^a_iU^0_j + \varphi_0\varphi_0\bar{g}^{ij}U^0_iU^0_j.$$

Hence if we set

$$\varphi_\infty = \frac{g^{ij}\varphi_i\varphi_j}{2\varphi_0} \tag{3.1}$$

and

$$U^0_0 U^\infty_\infty = 1,\ U^0_0 U^a_\infty = \bar{g}^{ij} U^a_i U^0_j,\ U^0_0 U^0_\infty = {}^1/_2\, \bar{g}^{ij} U^0_i U^0_j, \qquad (3.2)$$

we have

$$\bar{\varphi}_\infty = U^0_\infty \varphi_0 + U^i_\infty \varphi_i + U^\infty_\infty \varphi_\infty .$$

If we also set

$$U^\infty_\alpha = 0 \qquad (3.3)$$

and adopt the convention that Greek letters from the latter part of the alphabet run over the range $0, 1, \ldots, n, \infty$, we have

$$\bar{\varphi}_\sigma = U^\tau_\sigma \varphi_\tau . \qquad (3.4)$$

Conformal tensors are now defined as geometrical objects having $(n+2)^k$ components in each coördinate- and gauge-frame, and a law of transformation based in the tensorial manner upon the $(n+2)$-rowed matrix,[2]

$$\begin{array}{c|c|c} U^0_0 & U^0_i & U^0_\infty \\ \hline 0 & U^j_i & U^j_\infty \\ \hline 0 & 0 & U^\infty_\infty \end{array} \qquad (3.5)$$

For the inverse matrix we use V^σ_τ, so that

$$U^\sigma_\tau V^\tau_\omega = \delta^\sigma_\omega . \qquad (3.6)$$

The matrices of the form (3.4) form a pseudo group.[3]

The components of what may be regarded as the fundamental conformal tensor $g_{\sigma\tau}$ have appeared already in (3.1). The components g_{ij} have the significance given them in §1 and the other components are

$$g_{0\infty} = g_{\infty 0} = -1,\ g_{00} = g_{\infty\infty} = g_{0i} = g_{i0} = g_{\infty i} = g_{i\infty} = 0. \qquad (3.7)$$

This conformal tensor and its dual, $g^{\sigma\tau}$, defined in the usual way, will be used for raising and lowering conformal indices. For example, the equation $\varphi^\sigma = g^{\sigma\tau}\varphi_\tau$ includes the following:

$$\varphi^i = g^{ij}\varphi_j,\quad \varphi^0 = -\varphi_\infty,\quad \varphi^\infty = -\varphi_0 .$$

4. The law of transformation of the fundamental conformal tensor,

$$\bar{g}_{\sigma\tau} = g_{\lambda\mu} U^\lambda_\sigma U^\mu_\tau , \qquad (4.1)$$

summarizes in one formula the law of transformation (1.3) and the algebraic relations connecting the elements of the matrix (3.5). It of course does not include the condition

$$U^0_k = \frac{\partial U^0_0}{\partial x^k} . \qquad (4.2)$$

The law of transformation of the first derivatives of the g's is comprised in (1.5) which may now be rewritten in the form

$$\bar{\Gamma}^k_{ij} = V^k_\sigma \left(\Gamma^\sigma_{\tau b} U^\tau_i + \frac{\partial U^\sigma_i}{\partial x^b} \right) \frac{\partial x^b}{\partial \bar{x}^j} \tag{4.3}$$

if we set

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}, \Gamma^\infty_{ij} = g_{ij}, \Gamma^i_{\circ j} = \delta^i_j, \Gamma^\infty_{\circ j} = 0, \tag{4.4}$$

and leave $\Gamma^\circ_{\alpha i}$ and $\Gamma^\sigma_{\infty j}$ arbitrary.

A conformal connection may be defined as a geometric object with the law of transformation,

$$\bar{\Gamma}_i = V \left(\Gamma_j U + \frac{\partial U}{\partial x^j} \right) \frac{\partial x^j}{\partial \bar{x}^i}. \tag{4.5}$$

In (4.5) a matrix notation is used to replace the conformal indices. Thus Γ_j stands for $\Gamma^\sigma_{\tau j}$, etc. The equations (4.3) are a subclass of the equations symbolized in (4.5).

Conformal differentiation may now be defined in an obvious way. For instance, the covariant derivative of a contravariant conformal vector T is a geometric object $T_{,i} = \dfrac{\partial T}{\partial x^i} + \Gamma_i T$, whose law of transformation is

$$T_{,i} = V T_{,j} \frac{\partial x^j}{\partial \bar{x}^i} \tag{4.6}$$

The conformal derivative may be converted into a conformal tensor with two new antisymmetric indices by the method of Vanderslice,[4] namely,

$$T_{,\sigma\tau} = T_{,i} A^i_{\sigma\tau}, \tag{4.7}$$

where

$$A^i_{\sigma\tau} = \delta^{i\,\infty}_{\sigma\tau} \equiv \delta^i_\sigma \delta^\infty_\tau - \delta^i_\tau \delta^\infty_\sigma. \tag{4.8}$$

This makes it possible to repeat the operation of covariant differentiation indefinitely.

Those components of the fundamental conformal connection which were left undetermined in (4.4) may be fixed by imposing invariant conditions in such a way that (4.5) embodies the transformation laws not only of the g_{ij}'s and their first derivatives, but also of certain combinations of the second derivatives. The requisite formulas[5] are

$$\Gamma^\circ_{ij} = -\frac{1}{n-2} \left(R^k_{ijk} - \frac{g^{p\varepsilon} R^k_{p\varepsilon k}}{2(n-1)} g_{ij} \right) \tag{4.9}$$

in which R^i_{jkl} is formally the curvature tensor of the g_{ij}'s, and

$$\Gamma^i_{\infty j} = g^{ik}\Gamma^0_{kj}, \quad \Gamma^\infty_{\infty i} = \Gamma^0_{\infty i} = \Gamma^0_{0i} = 0. \tag{4.91}$$

At this point everything is in order for constructing a complete system of invariants by the methods[6] of T. Y. Thomas and Vanderslice.

5. The contravariant conformal vectors $(X^0, X^1, \ldots, X^n, X^\infty)$ at a point x of the underlying space may be represented as points of a projective space $P_{n+1}(x)$ of $n+1$ dimensions, the $n+2$ components being regarded as homogeneous coördinates. When the coördinate- and gauge-frame of the underlying space is transformed arbitrarily, these homogeneous coordinates undergo the transformation $\overline{X}^\sigma = V^\sigma_\tau X^\tau$. These transformations leave the quadratic form,

$$g_{\sigma\tau}X^\sigma X^\tau \equiv g_{ij}X^iX^j - 2X^0X^\infty, \tag{5.1}$$

invariant, and also the equation of the hyperplane $X^\infty = 0$ and the coordinates of the point $(\lambda, 0, \ldots, 0)$.

The quadric $Q_n(x)$ whose equation is $g_{\sigma\tau}X^\sigma X^\tau = 0$ will be referred to as the conformal tangent space at x. These conformal spaces correspond to the affine tangent spaces $A_n(x)$ in which the coördinates are the affine differentials $(dx^1, dx^2, \ldots, dx^n)$, by means of the equations,

$$\begin{aligned} Z^\infty &= \frac{\rho}{\varphi_0^2}\left(\varphi^\infty \frac{1}{2}g_{ij}dx^idx^j\right) \\ Z^k &= \frac{\rho}{\varphi_0^2}\left(\varphi^k \frac{1}{2}g_{ij}dx^idx^j + \varphi^\infty dx^k\right) \\ Z^0 &= \frac{\rho}{\varphi_0^2}\left(\varphi^0 \frac{1}{2}g_{ij}dx^idx^j + \varphi_i dx^i + \varphi^\infty\right) \end{aligned} \tag{5.2}$$

in which ρ is an arbitrary scalar and φ^σ an arbitrary but fixed contravariant conformal vector representing a point of $Q_n(x)$.

Since $\varphi^\infty = -\varphi_0$ it is clear that these formulas are a direct generalization (by adding the terms in the first column) of the formulas connecting affine and projective vectors. Geometrically, they represent a stereographic projection from the point φ^σ upon the plane $X^\infty = 0$. This projection makes a general point Z^σ of $Q_n(x)$ correspond to a point X^σ given by

$$X^\sigma = \frac{Z^\infty\varphi^\sigma - \varphi^\infty Z^\sigma}{\varphi^\infty}. \tag{5.3}$$

The quantities $X^\alpha (\alpha = 0, 1, \ldots, n)$ are homogeneous coördinates in the plane $X^\infty = 0$, and are clearly components of a projective vector. This projective vector determines an affine vector by means of the known formula

$$dx^i = \frac{\varphi_0 X^i}{\varphi_\alpha X^\alpha} \tag{5.4}$$

The hyperplane at infinity for these affine coördinates dx^i is the intersection of $\varphi_\sigma Z^\sigma = 0$ with $Z^\infty = 0$. The result of combining (5.3) with (5.4) is

$$dx^i = \frac{\varphi^\infty Z^i - Z^\infty \varphi^i}{\varphi_\sigma Z^\sigma} \tag{5.5}$$

and these equations are inverse to (5.2).

[1] O. Veblen, *Projektive Relativitätstheorie*, Berlin, 1933.

[2] This matrix is, of course, a modification of the one introduced by T. Y. Thomas in these PROCEEDINGS, **12**, 356 (1926).

[3] This pseudo group may be discussed in much the same way as what I called the enlarged conformal group in these PROCEEDINGS, **14**, 738 (1928).

[4] J. L. Vanderslice, these PROCEEDINGS, **20**, 674 (1934).

[5] These differ slightly from the formulas of T. Y. Thomas (see reference in Note 2) because of the difference in our matrices of transformation and because the second covariant index of our connection runs from 1 to n instead of from 0 to n as with Thomas. I am indebted to Dr. Vanderslice for making the computation.

[6] The references are given in Vanderslice's paper (Note 4 above).

SOME EXPANSIONS ASSOCIATED WITH BESSEL FUNCTIONS

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1. *An Expansion for the Product of Two Bessel Functions.*—1.1. An expansion for the product of two Bessel functions obtained by one of us[1] led to the discovery of a different expansion for the said product multiplied by the leading terms in the power series for the Bessel functions. Two proofs of this second expansion are given here. The first depends upon the fact that the function $V = J_a(u)u^a J_c(v)v^c$ satisfies the equation

$$u\frac{\partial^2 V}{\partial u^2} + (1-2a)\frac{\partial V}{\partial u} + uV + v\frac{\partial^2 V}{\partial v^2} + (1-2c)\frac{\partial V}{\partial v} + vV = 0$$

which is transformed by the substitutions $u = zx$, $v = z(1-x)$ into

$$z^2\frac{\partial^2 V}{\partial z^2} + (3-2r)z\frac{\partial V}{\partial z} + [z^2 - (2n+1)(2r+2n-1)]V$$

$$+ x(1-x)\frac{\partial^2 V}{\partial x^2} + [2s+2+(2r-3)x]\frac{\partial V}{\partial x} + (2n+1)(2r+2n-1)V = 0$$

where $r = a + c + 1/2$, $s = -a - 1/2$ and n is an arbitrary constant. A particular solution of the last equation is

$$V = V_n = z^{r-1} J_{r+2n}(z) D_x^{2n} [x^{2a+2n} (1 - x)^{2c+2n}].$$

If we assume that the former solution can be expressed as the sum of terms of type $A_n V_n$, where n runs from 0 to ∞, the constant A_n may be found by letting x approach zero and comparing our assumed series with Sonine's expansion[2]

$$z^{-s} J_{r+s}(z) = 2^{-s} \sum_{n=0}^{\infty} \binom{s}{n} \frac{(r + 2n)\, \Gamma(r + n)}{\Pi(r + s + n)} J_{r+2n}(z).$$

It is thus found that

$$J_a(u) u^a J_c(v) v^c = \sum_{n=0}^{\infty} A_n V_n$$

where

$$A_n = (-)^n \frac{(r + 2n)\, \Pi(r + n - 1)}{2^{2n-1/2}\, \Pi(a + n)\, \Pi(c + n) n!}.$$

1.2. An independent proof of the validity of the expansion is based upon the equation[1]

$$J_a(zx) J_c(z - zx) = x^a (1 - x)^c A \int_P J_{a+c}(zw) w^{-a-c} t^{-a-1} (1 - t)^{-c-1} dt$$

where

$$4\pi A = i + \cot \pi(a + c), \quad w^2 = \frac{x^2}{t} + \frac{(1 - x)^2}{1 - t}$$

and the contour P is the double loop $(1+, 0+, 1-, 0-)$ of the Pochhammer type. The arguments of t and $(1 - t)$ are zero at the starting point which lies on the real axis between the points $t = 0$ and $t = 1$. By using the expansion[2]

$$w^{-a-c} J_{a+c}(zw) = (z/2)^{a+c-b} \sum_{n=0}^{\infty} \frac{\Pi(b + n - 1)(b + 2n)}{n!\, \Pi(a + c)}$$

$$F(b + n, -n;\ a + c + 1;\ w^2) J_{b+2n}(z) \qquad (1)$$

which is valid when b and $a + c$ are not negative integers, and integrating termwise we obtain terms involving integrals of type

$$I = A \int_P F(b + n, -n;\ a + c + 1;\ w^2) t^{-a-1} (1 - t)^{-c-1} dt.$$

The series may be shown to converge uniformly with respect to t when b is not an integer and so the integration of the series is permissible. Indeed, when n is a positive integer,

$$\Gamma(b+n)F(b+n, -n; a+c+1; w^2)/[\Pi(a+c)\Gamma(b+n-a-c)] = A' \int_P (\tau - w^2)^n d\tau/[\tau^{a+c+n+1}(1-\tau)^{b+n-a-c}] \qquad (2)$$

where $A' = i + \cot \pi b$, as may be verified by expanding $(\tau - w^2)^n$. This relation defines the hypergeometric function for all values of w. Choosing the loops of the contour P to be circles of unit diameter with centers at 0, 1, respectively, we see that the total length of P is 4π and that on P we have the inequalities

$$|\tau - w^2| \leq \frac{3}{2} + |w^2|, \quad |\tau^{-a-c-n-1}| \leq 2^{n+1}B_1, \quad |(1-\tau)^{a+c-b-n}| \leq 2^n B_2$$

where $\log B_1 = |R(a+c)| \log 2 + 2\pi|I(a+c)|$, $\log B_2 = |R(b-a-c)| \log 2 + 2\pi|I(b-a-c)|$. Thus the modulus of the right-hand side of (2) is less than

$$|i + \cot(\pi b)| \left(\frac{3}{2} + |w^2|\right)^n 2^{2n+1} B_1 B_2$$

and so the modulus of the nth term in (1) is less than

$$\left| (b+2n)\Gamma(b+n-a-c)\ (i+\cot(\pi b))\ \left(\frac{3}{2} + |w^2|\right)^n 2^{2n+1} B_1 B_2 J_{b+2n}(z)/n! \right|$$

As $n \longrightarrow \infty$, $\Gamma(b+2n+1)J_{b+2n}(z) \longrightarrow (z/2)^{b+2n}$ and so the series converges uniformly for all finite values of w^2 unless b is an integer. When b is an integer further investigation is needed to establish the uniform convergence but this may be avoided by excluding this case for the present and then returning later to an examination of the final result by the method of continuity.

1.3. The integral I may be expressed as a hypergeometric function of x when $b = a + c + 1/2$. In order to carry through the proof we assume for the present that neither $a + c + 1/2$ nor $a + c$ is an integer. We first use Kummer's relation

$$F(p, q; p+q+1/2; 4y - 4y^2) = F(2p, 2q; p+q+1/2; y)$$

wherein $p = a + c + n + 1/2$, $q = -n$, $4y(1-y) = w^2$, $2y = 1 + i(t-x)t^{-1/2}(1-t)^{-1/2}$. The usual restrictions $|2y| < 1$, $|4y(1-y)| < 1$ are not necessary here since the series terminate and each side is merely a

polynomial in y. The resulting series is then arranged in ascending powers of x and the hypergeometric function in the integrand becomes

$$\sum_{r=0}^{2n} \frac{(2a + 2c + 2n + 1)_r\ (-2n)_r}{r!(a + c + 1)_r} \left(\frac{-ix}{2\sqrt{t(1-t)}}\right)^r A_r(t)$$

where

$$A_r(t) = F(2a + 2c + 2n + r + 1, r - 2n; a + c + r + 1; \frac{1}{2} + \frac{i}{2}\sqrt{t/(1-t)})$$

$$= (-)^n i^r [t/(1-t)]^{n-1/2r} F\left(\frac{1}{2}r - n, \frac{1}{2}r + \frac{1}{2} - n;\ a + c + r + 1;\ 1/t\right).$$

The last step may be verified by using Kummer's formula combined with the relation between hypergeometric functions of arguments y and $y/(y-1)$. It is convenient to perform the transformations at the point $t = N \exp i\pi$, $\arg(1-t) = 0$, where N is a large positive constant, and then extend the result to the contour P by analytical continuation. When written in full the expression for I is

$$I = (-)^n A \int_P dt \sum_{r=0}^{2n} \sum_{s=0}^{m} \frac{(2a+2c+2n+1)_r\ (-2n)_r\ (\frac{1}{2}r-n)_s\ (\frac{1}{2}r+\frac{1}{2}-n)_s x^r}{r!(a+c+1)_r 2^r s!(a+c+r+1)_s t^{a+r-n+s+1}(1-t)^{c+n+1}}$$

where $m \leq (n - \frac{1}{2}r)$. Using now Pochhammer's extension of the Eulerian integral of the first kind and summing the resultant hypergeometric series in s of argument unity by means of the formula of Gauss, we have finally

$$I = (-)^n \frac{\Pi(a + c)\Pi(a + n - \frac{1}{2})}{\Pi(c + n)} \sum_{r=0}^{2n} \frac{(2a + 2c + 2n + 1)_r\ (-2n)_r}{2^r \Pi(a + \frac{1}{2}r)\ \Pi(a + \frac{1}{2}r - \frac{1}{2})} \frac{x^r}{r!} =$$

$$(-)^n \frac{\Pi(a + c)\Pi(2a + 2n)}{2^{2n}\Pi(a+n)\Pi(c + n)\Pi(2a)} F(2a + 2c + 2n + 1, -2n;\ 2a + 1;\ x).$$

When this hypergeometric function is expressed as a Jacobi polynomial and the corresponding value of I is placed in the infinite series for $J_a(zx)$ $J_c(z - zx)$ we obtain the expansion given in section 1.1

$$x^a(1-x)^c J_a(zx)\ J_c(z - zx) = \sum_{n=0}^{\infty} (-)^n \frac{(a+c+2n+\frac{1}{2})\ \Pi(a+c+n-\frac{1}{2})}{2^{2n-\frac{1}{2}}\ \Pi(a + n)\ \Pi(c + n)n!}$$

$$z^{-1/2} J_{a+c+2n+1/2}(z) D_x^{2n}[x^{2a+2n}(1 - x)^{2c+2n}].$$

The restriction that $a + c$ be not an integer and that $a + c + \frac{1}{2}$ be not a positive integer may be removed by the use of continuity but the restriction that $a + c + \frac{1}{2}$ be not a negative integer is necessary. Using Lommel's expansion to compare the expansion for $J_a(zx)$ $J_c(z - zx)$, mentioned in

the introduction, with the expansion developed here we obtain the equation

$$F(-2k, a+c+1;\ a-k+1;\ x) = \sum_{n=0}^{k} C_n F(-2n, 2a+2c+2n+1;\ 2a+1;\ x)$$

where

$$C_n = \frac{(-)^{n+k}(a+c+2n+1/2)\Pi(2a+2c+2n)\Pi(a+n-1/2)\Pi(a-k)\Pi(c+k)(2k)!}{2^{2k+2c}\Pi(a+c+k+n+1/2)\Pi(c+n)\Pi(2a)\Pi(a+c)(2n)!\,(k-n)!}$$

2. *Generating Functions for Bessel's Functions of Non-Integral Order.*— If $R(\nu) > 0$,

$$e^{-x\tau}\int_0^1 \exp\left((y-yt)/\tau\right)t^{\nu-1}\,dt = e^{-x\tau}\sum_{r=0}^{\infty} y^r\tau^{-r}/(\nu)_{r+1} = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}(-x)^s\tau^{s-r}y^r/[s!(\nu)_{r+1}].$$

Since the double series is absolutely convergent we may write it in the form

$$\sum_{n=0}^{\infty}(y/\tau)^n\sum_{s=0}^{\infty}(-xy)^s/[s!(\nu)_{n+s+1}] + \sum_{n=1}^{\infty}(-x\tau)^n\sum_{r=0}^{\infty}(-xy)^r/[(\nu)_{r+1}(n+r)!]$$

where in the first series $n = r - s$ and in the second series $n = s - r$, consequently when $R(\nu) > 0$

$$\frac{e^{-x\tau}}{\Gamma(\nu)}\int_0^1 \exp\left((y-yt)/\tau\right)t^{\nu-1}\,dt = \sum_{n=0}^{\infty}(y/\tau)^n(xy)^{-n/2-\nu/2}\,J_{n+\nu}(2\sqrt{xy}) + \sum_{n=1}^{\infty}(-x\tau)^n F(1;\ n+1,\ \nu+1;\ -xy)/[n!\Gamma(\nu+1)].$$

If we multiply both sides of this equation by $\exp(-ux)\,x^{m-1}$, where $R(u)$, $R(m)$, $R(u+\tau) > 0$, and integrate between $x = 0$ and $x = \infty$ we obtain a convergent integral on the left. Moreover, when $|\tau| < |u|$ the series on the right may be integrated term by term and the factor $(u+\tau)^m$ arising from integration on the left may be expanded by the binomial theorem. Upon arranging the left-hand side as a Laurent series in τ and comparing it with the expression on the right we find

$$\frac{y^{\frac{n+\nu}{2}}\Gamma(m)}{u^m\Gamma(n+\nu+1)}F(m;\ n+\nu+1;\ -y/u) = \int_0^{\infty} e^{-xu}x^{m-\frac{n+\nu}{2}-1}\,J_{n+\nu}(2\sqrt{xy})\,dx,$$

$$I^m_{n,\nu}(u) \equiv \int_0^{\infty} e^{-xu}x^{m+n-1}F(1;\ n+1,\ \nu+1;\ -xy)\,dx$$

$$= u^{-m-n}\Gamma(m+n)F(1, m+n;\ n+1, \nu+1;\ -y/u). \tag{3}$$

The first integral is well known and the second may be readily verified when $R(u) > 0$, by expanding the hypergeometric function, and integrating termwise.

When $u = 0$ the integral $I^m_{n,\nu}$ converges only if $0 < m + n < 1$, as may be seen by Barnes' asymptotic formula for ${}_1F_2$, assuming y real and positive. If, in addition, $m + n - \nu$ is a positive integer it may be shown that the integral is zero by letting $u \longrightarrow 0$ in the hypergeometric expression for $I^m_{n,\nu}$. The general value of Hardy's type[5] thus obtained is also the actual value since the integral converges. To do this it is convenient to use the reduction formula

$$(m+n-1)yF(1, m+n;\ n+1, \nu+1;\ -y/u) = n\nu u[1 - F(1, m+n-1;\ n,\nu;\ -y/u)]$$

to obtain

$$yI^m_{n,\nu} = n\nu u^{1-m-n}\Gamma(m+n-1)F(1, 1-n, 1-\nu; 2-m-n; u/y) + y^{1-n}u^{-m}n!\Gamma(m)(n-\nu-1)_n F(m;\ \nu-n+1; -y/u) \tag{4}$$

This is merely the result of a formal transformation of the function ${}_2F_2$ and holds without restriction. It is, in fact, a particular case of Barnes' asymptotic formula[4] for $F(a, c;\ u, v;\ z)$ which holds when $|\arg z| < 3\pi/2$. Making use of Kummer's relation

$$F(m;\ \nu-n+1;\ -y/u) = \exp(-y/u)F(\nu-n-m+1;\ \nu-n+1;\ y/u)$$

we note that the confluent hypergeometric function on the right is a polynomial when $n + m - \nu$ is a positive integer. The last term in (4) then becomes zero as $u \longrightarrow 0$. If $m + n < 1$ the first term in (4) also tends zero and so in these circumstances the actual value of $I^m_{n,\nu}$ is zero.

This result may be compared with Orr's formula[5]

$$\int_0^\infty \lambda^{m-1}F(a_1, a_2, \ldots, a_n;\ c_1, c_2, \ldots, c_n;\ -\lambda)d\lambda = \Gamma(m)\prod_{r=1}^{n}\frac{\Gamma(c_r)\ \Gamma(a_r-m)}{\Gamma(a_r)\ \Gamma(c_r-m)}$$

$(m > a_l$ where a_l is the algebraical least $a_r)$ and with the result of his remark that by putting $\lambda = z/a_1$ and making $a_1 \longrightarrow \infty$, a corresponding formula may be derived for the case in which the number of the a's is less than the number of c's.

Several definite integrals which partake of the nature of generating functions are

$$\int_0^K J_\mu(z\,dn\,u)sn^{2\nu-1}u\,cn^{2\rho-1}u\,dn^{1+\mu}u\,du = \sum_{n=0}^{\infty}k^{2n}(-)^n\left(\frac{z}{2}\right)^n\frac{B(\nu+n,\rho)}{n!2}J_{\mu-n}(z)$$

$$\int^{K} J_\mu(z\,dn\,u)sn^{2\nu-1}u\,cn^{2\rho-1}u\,dn^{1-\mu}u\,du=\sum_{n=0}^{\infty} k^{2n}\left(\frac{z}{2}\right)^n\frac{B(\nu+n,\rho)}{n!2}\,J_{\mu+n}(z)$$

$$\int^{1} J_\mu(zx)x^{\mu+1}\exp[{}^1/_2tz(1-x^2)]dx=\sum_{n=0}^{\infty} t^n\,z^{-1}\,J_{\mu+n+1}(z)$$

$$\int^{1} J_\mu(zx)x^{\mu+1}\exp\,[{}^1/_2t(1-x^2)]\,dx=\sum_{n=0}^{\infty} t^n\,z^{-n-1}\,J_{\mu+n+1}(z).$$

In the first two expressions the Bessel functions of the first kind (J_ν's) may be replaced throughout by functions of the second kind (Y_ν's); and further, if the factor $(-)^n$ in the summation involving $J_{\mu-n}(z)$ be omitted the functions of the first kind may be replaced throughout by modified functions of the second kind (K_ν's). These equations, which hold for $R(\nu)$, $R(\rho)>0$, may be verified by replacing $z\,dn\,u$ by $\sqrt{z^2-z^2k^2sn^2u}$, expanding the Bessel functions in the integrand, using Lommel's expansion, and integrating termwise.

The last two equations, which hold when $R(\mu)>-1$ and which may be verified by using[6]

$$J_{\mu+n+1}(z)=\frac{z^{n+1}}{2^n n!}\int_0^1 J_\mu(zx)x^{\mu+1}\,(1-x^2)^n dx,$$

have probably been noticed before. The analogous series $\sum_{n=0}^{\infty}t^n z^n J_{n+\nu}(z)/n!$ has already been summed by W. Kapteyn[7] and some other Dutch mathematicians.

[1] S. O. Rice, "On Contour Integrals for the Product of Two Bessel Functions," *Quart. Jour. Math.*

[2] G. N. Watson, *Theory of Bessel Functions*, §5.21.

[3] *Ibid.*, §5.23.

[4] E. W. Barnes, *Trans. Roy. Soc. Lond.*, **206**, Part XI, 295 (1906).

[5] G. H. Hardy, *Trans. Camb. Phil. Soc.*, **21**, 13 (1908); W. McF. Orr, *Ibid.*, **17**, 171 (1898); **19**, 151 (1900).

[6] G. N. Watson, l. c., §12.11.

[7] W. Kapteyn, *Vraagstuk*, **132**, Solutions by G. R. Boogaerdt't, S. C. van Veen and others. *Wiskundige Opgaven*, **14**, 269–271 (1928).

SIMIMYS, A NEW NAME TO REPLACE EUMYSOPS WILSON, PREOCCUPIED.—A CORRECTION

BY ROBERT W. WILSON

A description of a new genus of cricetine-like rodent was published in the January, 1935, issue of the PROCEEDINGS of the National Academy of

Sciences. T. S. Palmer has kindly brought to my attention the fact that the name *Eumysops*, proposed for the genus, is preoccupied by *Eumysops* Ameghino 1888. Hence, the new name *Simimys* is proposed for *Eumysops* Wilson.

Genus Simimys, new name

1935. *Eumysops* Wilson, *Proc. Nat. Acad. Sci.*, Vol. 21, No. 1, 26–32, Jan., 1935.

(Not *Eumysops* Ameghino, Lista de los Mamíf. Fós. de Monte Hermoso, 5–6, June, 1888).

Type species—*Eumysops simplex* Wilson, *Proc. Nat. Acad. Sci.*, Vol. 21, No. 1, 26–28, plate 1, figure 1, 1935.